Special Case

If A is a triangular matrix, then its determinant is the product of the entries on the main diagonal

Ex.
$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{pmatrix} \qquad \det(A) = 2 \times -2 \times 1 \times 3 = -12$$

Evaluation of a determinant using elementary operations

Let A and B be square matrices

① If B is obtained from A by interchanging 2 rows of A, then
$\det(B) = -\det(A)$

e.g.
$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$
$\det(A) = -6$
$\det(B) = 6$

② If B is obtained from A by adding a multiple of row of A to another row of A then $\det(B) = \det(A)$

③ If B is obtained from A by multiplying a row of A by a nonzero constant (c) then $\det(B) = c\det(A)$

Find $\det(A)$, $A = \begin{pmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \ominus \begin{pmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{pmatrix}$

$\xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{*} (-7) \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{R_3 + R_2} (-7) \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

$|A| = (-7)(1 \times -1 \times -1) = -7$

* 7 is taken as common factor

Ex Find det(A) $A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{pmatrix} \xrightarrow[R_4 - 3R_1]{R_2 - 4R_1,\ R_3 + 2R_1} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\det(A) = (1 \times 2 \times -3 \times -1) = 6$

Determinants and elementary column operations

Theorem, If A is an $n \times n$ matrix then $\det(A^T) = \det(A)$

* Rules are the same as the previous row rules

Ex evaluate

$$\begin{vmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} \quad C_1 \longleftrightarrow C_2 \quad \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix} \quad \det = -(1 \times 4 \times 2) = -8$$

Ex Find $|A|$

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{pmatrix} \xrightarrow[\text{common factor from } C_2]{\text{take } -2} -2 \begin{pmatrix} -1 & -1 & 2 \\ 3 & 3 & 4 \\ 5 & 5 & -3 \end{pmatrix} \xrightarrow{C_2 - C_1} -2 \begin{pmatrix} -1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 0 & -3 \end{pmatrix}$$

det(A) = Zero

Conditions that produce a zero determinant

1. An entire row/column consists of zeros
2. Two rows/columns are equal
3. One row/column is a multiple of another row/column

Ex. Evaluate $|A|$

$$A = \begin{pmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_5 - R_2]{R_4 + R_2} \begin{pmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ -3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{C_1 - 3C_4} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \xrightarrow{R_1 + R_2} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} = 5\,\big((-8 \times 5) - (-1 \times 13)\big)$$

$$= -135$$

## properties of determinants

① If $A$ and $B$ are square matrices of order $n$ then $\det(AB) = \det(A) \cdot \det(B)$

Note: $\det(A+B) \neq \det(A) + \det(B)$

② If $A$ is an $n \times n$ matrix and $(c)$ is a scalar then $\det(cA) = c^n \det(A)$

eg $A = \begin{pmatrix} 30 & 20 & 50 \\ 80 & 40 & 30 \\ 20 & 10 & 60 \end{pmatrix}_{3\times 3}$ $\quad A = 10 \begin{pmatrix} 3 & 2 & 5 \\ 8 & 4 & 3 \\ 2 & 1 & 6 \end{pmatrix}$, $\begin{vmatrix} 3 & 2 & 5 \\ 8 & 4 & 3 \\ 2 & 1 & 6 \end{vmatrix} = -21$

$\det(A) = 10^3 \times -21 = -21000$

③ A square matrix is invertible if and only if its determinant $\neq 0$

④ If $A$ is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

proof

$A A^{-1} = I$

$|A A^{-1}| = |I|$

$|A| \cdot |A^{-1}| = 1$

$|A^{-1}| = \frac{1}{|A|}$

⑤ $\det(A^{T}) = \det(A)$

<u>Equivalent Conditions for a non-singular matrix</u>

If $A$ is non singular then:

1- $A$ is invertible

2- $\det(A) \neq 0$

3- $Ax = b$ has a unique Solution

4- $Ax = 0$ has only the trivial solution (where variables are all zeros)

Ex Which of the following has unique solution

a)

$$\begin{aligned} 2x_2 - x_3 &= 1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix}$$

$C_3 = \frac{C_2}{-2} \Rightarrow \det = 0$

does not have unique solution

b)

$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

$$\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$\det \neq 0$

has unique Solution

<u>Applications of determinants</u>

1- Finding inverse of a matrix

$$A^{-1} = \frac{1}{|A|} \cdot \text{adjoint}(A)$$

$$\text{Adjoint matrix} = (\text{Cofactor matrix}^{*})^{T}$$

Ex Find the inverse of A

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$\text{Cofactor matrix }(A) = \begin{pmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{pmatrix} \quad \text{Adjoint matrix }(A) = \begin{pmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\det|A| = 3$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 6 & 7 \\ 1 & 6 & 1 \\ 2 & 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{pmatrix}$$

2- Cramer's Rule

<u>Theorem</u>

The unique Solution of $Ax = b$ has entries given by $x_i = \dfrac{\det A_i(b)}{\det(A)}$

Where $A_i(b)$ is the matrix obtained from A by replacing column $i$ by the vector $b$

Ex Find $X_1$ given the following system

$$5x_1 + x_2 - x_3 = 4$$
$$9x_1 + x_2 - x_3 = 1$$
$$x_1 - x_2 + 5x_3 = 2$$

$$A = \begin{pmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{pmatrix} \qquad \det(A) = -16$$

$$A_1 = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{pmatrix}$$

first column in A is replaced by the column of b

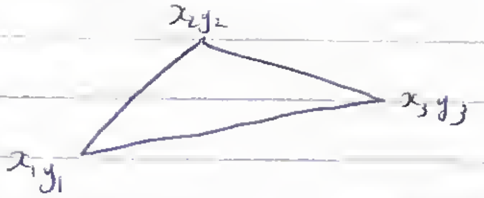
$\det A_1 = 12$

$$x_1 = \frac{12}{-16} = -\frac{3}{4}$$

## Applications of determinants in geometry

1. Area of a triangle in a plane

$$\text{Area} = \pm \frac{1}{2} \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$




* $\ominus$ sign is used to change the sign of determinant if it was -ve

2. Test of collinearity in plane

$$\det = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$(x_1, y_1)$
$(x_2\ y_2)$
$(x_3\ y_3)$

3. Equation of a line in plane

$(x_1 y_1)$ , $(x_2 y_2)$

$$\det = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \text{equation of straight line}$$

$y = mx + c$

or $ax + by = c$

4. Volume of tetrahedron

$$\text{Volume} = \pm \frac{1}{6} \det \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix}$$